Sydney Technical High School

HSC Assessment Task 1

December 2011

Mathematics

Time allowed — 70 minutes

Instructions

- Use a blue or black pen.

- Approved calculators may be used.

- All necessary working must be shown. Marks may not be awarded for careless or badly arranged work.

- Total marks — 55

- Attempt all questions, using the booklets provided.

- Marks awarded are shown on each question.
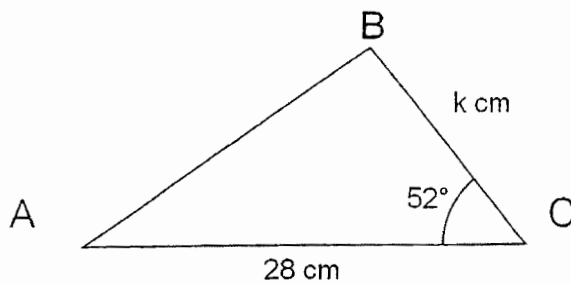
- Start each question on a new page.

Question 1 8 Marks

a) Solve $\cos x = -\frac{1}{\sqrt{2}}$ for $0° \leq x \leq 360°$ 2

b) Given $\tan \theta = \frac{-5}{12}$, find $\sin \theta$, if θ is obtuse. 1

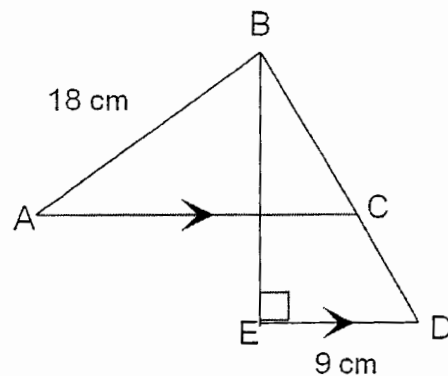c) The area of the triangle is 253.74 cm² and $\angle ACB = 52°$. 2

Calculate the value of k to the nearest centimetre.

d) In the diagram, $AB = 16$ cm, $AC = 20$ cm, $BE = 15$ cm and $ED = 9$ cm.
$BE \perp ED$ and $AC \parallel ED$.

    (i)   Find the size of $\angle BDE$, to the nearest degree. 1

    (ii)   Using the Sine Rule, find the size of $\angle ABC$, to the nearest degree. 2
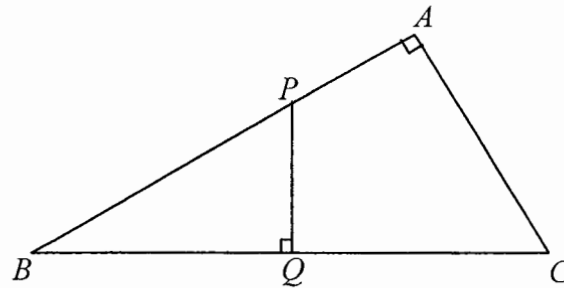
a) A point P(x,y) moves so that the sum of the squares of its distance from   **3**
   each of the points A(0,0) and B(4,0) is equal to 40.

   Show that the locus of P(x,y) is a circle, and state its radius and centre.
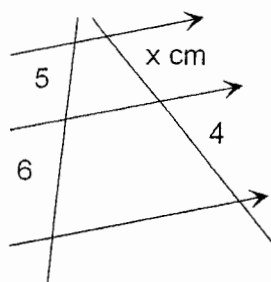
b)

   In the diagram, ABC is a right angled triangle with $AB = 8$ cm and $AC = 6$ cm.

   If the triangle is folded along the line PQ, vertex B coincides with vertex C.

   (i) Show that triangles ABC and QBP are similar.   **2**

   (ii) Find the length of PQ.   **1**

c) Find the value of x in the diagram. (Give a reason)   **2**

Question 3   **8 Marks**

a) The first three terms of a sequence are *7, 13* and *19*.

     (i)   Is this sequence arithmetic, geometric or neither? (Show reasoning)   1

     (ii)   Find a simplified expression for the *nth* term, T_n.   1

     (iii) Find T_{32}.   1

     (iv) Is *243* a term of the sequence? (Show appropriate working.)   2

b)   Find the values of *A, B* and *C* if $3x^2 + 5x \equiv A(x+1)^2 + B(x+1) + C$ 3

Question 4   **7 Marks**

a)   (i)   The fourth term of a geometric sequence is $\frac{2}{9}$ and the seventh term is $\frac{2}{243}$.   3
            Find the first three terms.

    (ii)   Explain why a limiting sum exists. (Show reasoning)   1

    (iii) Find the limiting sum (sum to infinity).   1

b)   Find the geometric series whose first term is 3 and whose limiting sum is 9.   2

Question 5 9 Marks

a) The equation of a parabola is given as $(x - 2)^2 = 4y$.

   (i) Find the coordinates of the focus. 1

   (ii) What is the equation of the directrix? 1

   (iii) Sketch the parabola, showing all important features. 2

b) Find the equation of the parabola with vertex $(-1,2)$, axis parallel to the x axis and passing through the point $(7,10)$. 2

c) Find the equation of the tangent to the parabola $x^2 = -16y$ at the point $(-2, -\frac{1}{4})$. 3

Question 6 8 Marks

a) For the parabola with equation $y = 10 + 3x - x^2$, find: 4

   (i) the equation of the axis of symmetry.

   (ii) the coordinates of the vertex.

   (iii) the x intercepts.

   (iv) the values of x for which $10 + 3x - x^2 > 0$.

b) Write a quadratic equation with roots α and β if $\alpha + \beta = -2$ and $\alpha\beta = 6$. 1

c) Make a suitable substitution and solve $(x^2 - 2)^2 - 7(x^2 - 2) = 0$. 3

a)   In the diagram, *PQRS* is a parallelogram. *QR* is produced to *U* so that *QR* = *RU*.
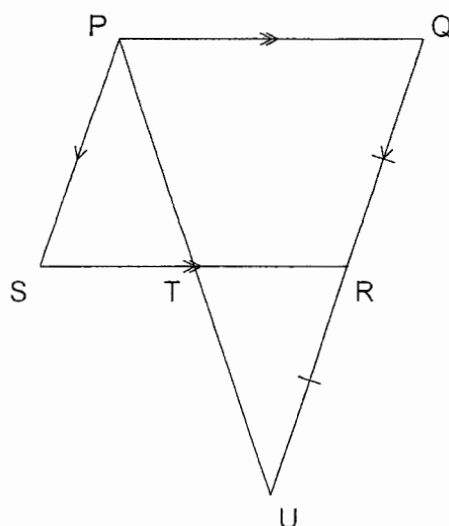
(i)   Giving appropriate reasons, prove that *ΔPST* and *ΔURT* are congruent.   3

(ii)   Hence or otherwise, show that *T* is the midpoint of *SR*.   1

b)   Two vertical poles of height *10m* and *15m* are *8m* apart. Wire stretches from the top of each pole to the foot of the other. See diagram below.
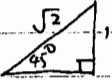
(i)   Use the properties of similar figures to find an expression for *w* in   1
        terms of *h*. [Hint: Let *DC* = *w*]

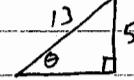(ii)   Find the height, *h m* above the ground where the wires cross.   2
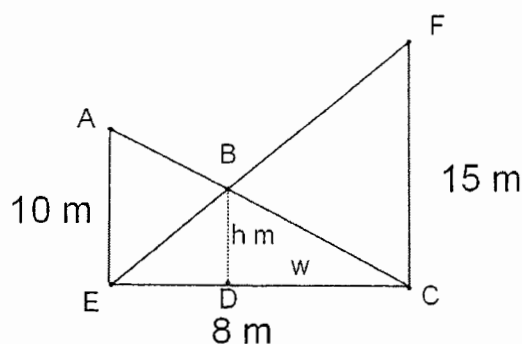
End of Paper

Q1 a) $\cos x = -\dfrac{1}{\sqrt{2}}$

∴ $x = 135°, 225°$

b) $\tan\theta = \dfrac{-5}{12}$

∴ $\sin\theta = \dfrac{5}{13}$

c) $A = \tfrac{1}{2}ab\sin C$

∴ $253.74 = \tfrac{1}{2}\times 28 \times k \times \sin 52°$

$\qquad = 14k\sin 52°$

∴ $k = \dfrac{253.74}{14\sin 52°}$

$\qquad ≑ 23\ cm$

d)(i) $\tan B\hat{D}E = \dfrac{15}{9\tfrac{2}{3}}$

$\qquad = \dfrac{15}{1\tfrac{1}{3}}$

∴ $B\hat{D}E ≑ 59°$

(ii) ∴ $A\hat{C}B = 59°$ (corresponding ∠s)

∴ $\dfrac{\sin A\hat{B}C}{20} = \dfrac{\sin 59°}{18}$

∴ $\sin A\hat{B}C = \dfrac{10\sin 59°}{9}$

$\qquad = 72°15'$

$\qquad ≑ 72°$

Q2 a) $(x-0)^2 + y^2 + (x-4)^2 + y^2 = 40$

∴ $x^2+y^2+x^2-8x+16+y^2=40$

∴ $2x^2+2y^2-8x=24$

∴ $x^2+y^2-4x=12$

∴ $(x-2)^2+y^2=16$

circle: centre $(2,0)$

$\qquad$ radius 4

b)
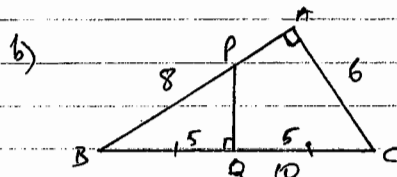
(i) In Δs QBP and ABC,

$A\hat{B}C$ is common

$B\hat{Q}P = B\hat{A}C = 90°$

∴ $\triangle QBP \ \|\|\| \ \triangle ABC$ (AAA)

(ii) $\dfrac{PQ}{BQ} = \dfrac{6}{8}$

∴ $PQ = \dfrac{5\times 6}{8}$

$\qquad = 3.75\ cm$

c) $\dfrac{x}{4} = \dfrac{5}{6}$

∴ $6x = 20$

∴ $3x = 10$

∴ $x = 3\tfrac{1}{3}\ cm$

Intercepts in proportion across parallel lines.

Q3

$\qquad\qquad$ For AP $\qquad\qquad$ For GP

a) (i) $13-7=6 \qquad\qquad \dfrac{13}{7}=1\tfrac{6}{7}$

$\qquad 19-13=6 \qquad\qquad \dfrac{19}{13}=1\tfrac{6}{13}$

$\qquad$ ∴ AP $\qquad\qquad\qquad$ ∴ Not GP

(ii) $T_n = a+(n-1)d$

$\qquad = 7+6(n-1)$

$\qquad = 6n+1$

(iii) ∴ $T_{32} = 6\times 32+1$

$\qquad = 193$

(iv) Let $T_n = 243$

∴ $6n+1 = 243$

∴ $6n = 242$

∴ $n = \dfrac{242}{6}$

$\qquad = 40\tfrac{1}{3}$

243 is not a term as $40\tfrac{1}{3} \notin \mathbb{N}$

b) $3x^2+5x \equiv A(x+1)^2 + B(x+1) + C$

$RHS = A(x+1)^2 + B(x+1) + C$

$\qquad = A(x^2+2x+1) + Bx+B+C$

$\qquad = Ax^2+2Ax+A+Bx+B+C$

$\qquad = Ax^2+(2A+B)x+(A+B+C)$

∴ $A=3$

$2A+B=5 \qquad\qquad A=3$

∴ $B=-1 \qquad\qquad B=-1$

$A+B+C=0 \qquad\quad C=-2$

∴ $3-1+C=0$

∴ $C=-2$

4b) $a=3$

$\dfrac{a}{1-r}=9$

∴ $\dfrac{3}{1-r}=9$

∴ $\dfrac{1}{1-r}=3$

∴ $1-r=\tfrac{1}{3}$

∴ $r=1-\tfrac{1}{3}$

$\qquad = \tfrac{2}{3}$

seq ⇒ $3, 2, \tfrac{4}{3}, \ldots$

Q4 a) (i) $T_n = ar^{n-1}$

∴ $T_4 = ar^3 = \tfrac{2}{9}$

$\qquad T_7 = ar^6 = \tfrac{2}{243}$

∴ $\dfrac{ar^6}{ar^3} = \dfrac{2}{243}\times\dfrac{9}{2}$

∴ $r^3 = \tfrac{1}{27}$

∴ $r = \tfrac{1}{3}$

$\dfrac{9}{27} = \dfrac{2}{9}$

∴ $a = \dfrac{2\times 27}{9}$

$\qquad = 6$

seq ⇒ $6, 2, \tfrac{2}{3}$

(ii) Limiting sum exists because $-1<r<1$ [or $|r|<1$]

(iii) $S_\infty = \dfrac{a}{1-r}$

$\qquad = \dfrac{6}{1-\tfrac{1}{3}}$

$\qquad = 9$

Q5 a) $(x-2)^2 = 4y$

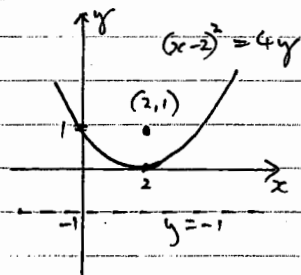General form $\Rightarrow (x-2)^2 = 4a(y-y_1)$ (iii)

$\therefore a=1, \; x_1=2, \; y_1=0$

vertex $(2,0)$

(i) Parabola upright $\therefore$ focus $(2,1)$

(ii) Directrix $y = -a$

$\therefore y = -1$

$(x-2)^2 = 4y$
$(2,1)$
$y=-1$

b) $(y-y_1)^2 = 4a(x-x_1)$

$\therefore x_1 = -1$

$y_1 = 2$

$\therefore (y-2)^2 = 4a(x+1)$

But $(7,10)$ satisfies

$\therefore (10-2)^2 = 4a(7+1)$

$64 = 4a \times 8$

$\therefore 4a = 8$

$\therefore a = 2$

$\therefore (y-2)^2 = 8(x+1)$

c) $x^2 = -16y$ $\quad (-2, -\frac{1}{4})$

$y = -\dfrac{x^2}{16}$

$\therefore \dfrac{dy}{dx} = -\dfrac{2x}{16}$

$= -\dfrac{x}{8}$

$= \dfrac{1}{4}$ when $x = -2$

$\therefore$ equation of tangent $\Rightarrow$

$y - y_1 = m(x - x_1)$

$y + \frac{1}{4} = \frac{1}{4}(x+2)$

$\therefore 4y + 1 = x + 2$

$\therefore x - 4y + 1 = 0$

Q6 a) $y = -x^2 + 3x + 10$

(i) axis of symmetry when $x = -\dfrac{b}{2a}$

$\therefore x = \dfrac{-3}{2 \times -1}$

$\therefore x = \dfrac{3}{2}$

(ii) $x = \dfrac{3}{2}$

$\therefore y = -\left(\frac{3}{2}\right)^2 + 3 \times \frac{3}{2} + 10$

$= -\frac{9}{4} + \frac{9}{2} + 10$

$= 12\frac{1}{4}$

$\therefore$ Vertex $\Rightarrow \left(\frac{3}{2}, 12\frac{1}{4}\right)$

(iii) $-x^2 + 3x + 10 \leq 0$

$\therefore x^2 - 3x - 10 = 0$

$\therefore (x-5)(x+2) = 0$

$\therefore x = -2, 5$

(iv) $a < 0$ $\therefore$ concave down

$\therefore$ positive when $-2 < x < 5$

Q6 b) $(x+\alpha)(x+\beta) = x^2 + (\alpha+\beta)x + \alpha\beta$

$\therefore x^2 + 2x + 6 = 0$ as $\alpha + \beta = -2$

$\alpha\beta = 6$

c) $(x^2-2)^2 - 7(x^2-2) = 0$

Let $w = x^2 - 2$

$\therefore w^2 - 7w = 0$

$\therefore w(w-7) = 0$

$\therefore w = 0, 7$

$\therefore x^2 - 2 = 0, 7$

$\therefore x^2 = 2, 9$

$\therefore x = \pm\sqrt{2}, \pm 3$

Q7 a) (i) In $\triangle PST$ and $\triangle URT$,

$\quad P\hat{T}S = U\hat{T}R$ (vertically opposite $\angle$s)

$\quad P\hat{S}T = U\hat{R}T$ (alternate $\angle$s on parallel lines)

$\quad QR = RU$ (data)

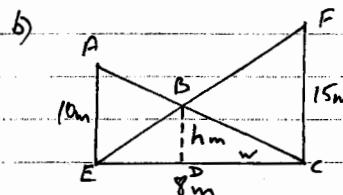$\quad SP = QR$ (opposite sides of parallelogram)

$\therefore RU = SP$

$\therefore \triangle PST \equiv \triangle URT$ (AAS) QED

(ii) $ST = RT$ (corresponding sides in congruent $\triangle$s)

$\therefore T$ is the midpoint of SR QED

b)

Let $DC = w$

(i) Now $\dfrac{w}{h} = \dfrac{8}{10}$

$\therefore w = \dfrac{4h}{5}$

(ii) $\dfrac{8-w}{h} = \dfrac{8}{15}$

$\therefore \dfrac{8 - \frac{4h}{5}}{h} = \dfrac{8}{15}$

$\therefore 8h = 15\left(8 - \frac{4h}{5}\right)$

$= 120 - 12h$

$\therefore 20h = 120$

$\therefore h = 6m$

It is interesting that if we change the distance between the poles, the value of h does not change. Try replacing the 8 with a variable (say d) to prove this.